Poisson's equation for Markov chains

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Outline

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2. Lyapunov bounds on a solution of Poisson's equation
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Discrete-time Markov chains

Let $\mathbf{\Phi} = \{\Phi_n, n \geq 0\}$ be a time-homogeneous discrete-time Markov chain (DTMC), evolving on a complete separable metric space E, whose Borel σ-algebra shall be dented by $\mathcal{B}(E)$.

The transition kernel is denoted by $P = (P(x, dy) : x, y \in E)$. Suppose that P is Ψ-irreducible and Harris positive recurrent with the unique stationary distribution $\pi = (\pi(dx) : x \in E)$.

Poisson's equation for DTMCs

Poisson's equation has the following form:

$$(I - P)\tilde{g} = \overline{g},$$

where I is the identity operator, $\overline{g}(x) = g(x) - \pi(g)$.

The function g is called the forcing function and is assumed to satisfy $\pi(|g|) < \infty$. The function $\tilde{g}$ is called a solution of Poisson's equation.

Uniqueness and existence

By Glynn & Meyn (1996), we know:

Uniqueness: If $\tilde{g}_1$ and $\tilde{g}_2$ are two solutions of Poisson's equation with $\pi(|\tilde{g}_1| + |\tilde{g}_2|) < \infty$, then $\tilde{g}_1(x) = c + \tilde{g}_2(x)$ for some c.

Existence: A commonly used solution is given by

$$\tilde{g}_{\alpha}(x) = \mathbb{E}_x\left[\sum_{k=0}^{\tau_\alpha - 1} \overline{g}(\Phi_k)\right], \; x \in E,$$

where α is an atom in $\mathcal{B}(E)$, and $\tau_\alpha = \inf\{k \geq 1 : \Phi_k \in \alpha\}$.

Connection-CLT

Under some conditions, a central limit theorem (CLT) holds (Meyn & Tweedie 2009), i.e.

$$n^{-\frac{1}{2}} \sum_{k=1}^{n} \overline{g}(\Phi_k) \Rightarrow N(0, \sigma^2(\boldsymbol{g})), \quad as\ n \to \infty,$$

and the variance constant $\sigma^2(\boldsymbol{g})$ is given by

$$\sigma^2(g) = 2\mathbb{E}_\pi[\tilde{g}(\Phi_0)\overline{g}(\Phi_0)] - \mathbb{E}_\pi[\overline{g}^2(\Phi_0)].$$

Connection-Hoeffding's inequality

▷ Hoeffding's inequality (Glynn & Ormoneit 2002, Choi & Li 2019, L. & Liu 2020):

If Poisson's equation admits a solution $\tilde{g}$ such that $\|\tilde{g}\|_\infty < \infty$, then for any $\varepsilon > 0$ and $n > 2\|\tilde{g}\|_\infty/\varepsilon$,

$$\mathbb{P}_x\left(\frac{1}{n}\sum_{k=1}^{n} g(\Phi_k) - \pi(g) \geq \varepsilon\right) \leq \exp\left\{\frac{-(n\varepsilon - 2\|\tilde{g}\|_\infty)^2}{2n\|\tilde{g}\|_\infty^2}\right\}.$$

Connection-perturbation analysis

▷ Perturbation theory (Glynn & Meyn 1996, L. 2015):
Let P and $\tilde{P}$ be positive recurrent Markov chains with invariant probability distributions π and ν respectively.

$$(\nu - \pi)g = \nu\Delta\tilde{g},$$

where $\Delta = \tilde{P} - P$ is the perturbation.

▷ Truncation approximations to invariant distribution (Masuyama 2016, L. & Li 2018):

$$({}_{(n)}\pi - \pi)g =_{(n)} \pi\Delta\tilde{g}.$$

Further connections of Poisson's equation

- ▷ Markov decision theory: Poisson's equation is known as the dynamic programming equation, and the functions $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$ are called the cost function and the value function (Guo & Hernández-Lerma 2009).
- ▷ MCMC algorithms: variance reduction problems (Mijatović & Vogrinc 2019).
- ▷ Ergocity for single-birth process with specific g (Chen & Zhang 2014).

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Concern problems

Solve the Poisson equation: *QBD* processes: Dendievel, L. & Latouche(2013); Bini et al. (2016) (General solution); Liu L. & Zhao (2023) (matrix-analytical method for countable chains).

Derive bounds on the solution: Glynn & Meyn (1996); Wu (2009).

Approximate the solution: Mijatović & Vogrinc (2019); Liu, L. & Zhao (2022) (Augmented truncation).

▷ In this talk, we present bounds and monotonicity about a solution of Poisson's equation.

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Petite set

Recall that a set $C \in \mathcal{B}(E)$ is called a petite set if there exists a positive constant λ, probability distributions φ and $a = (a(n), n \in \mathbb{Z}_+)$ such that

$$\sum_{n=0}^{\infty} a(n) P^n(x, \cdot) \geq \lambda \varphi(\cdot), \quad x \in C.$$

▷ C is called a ν_m-small set if $a(n) = 1_{\{n=m\}}$.

Lyapunov drift condition

Drift condition for f-ergodicity:

D(V,f,b,C): There exists a positive constant $b < \infty$, a set C and finite functions $V \geq 1$ and $f \geq 1$ such that

$$PV(x) \leq V(x) - f(x) + b \cdot \mathbf{1}_C(x), \quad x \in E,$$

where $\mathbf{1}_C(\cdot)$ is the indictor function in the set C.

▷ If $f = \beta V$ for some $0 < \beta < 1$, then it responds to geometric ergodicity.

▷ If $f = V^\alpha$ for some $0 < \alpha < 1$, then it responds to polynomial ergodicity.

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Bounds on a solution

Theorem 1.(Glynn, Lin & L. 2023)

Suppose that D(V,f,b,C) holds for a petite set C. Then for any measurable function g satisfying $0 \leq g \leq f$, there exists a solution $\tilde{g}$ such that

$$\frac{-b}{\inf_{x\in E} f(x)}(V(x)+bd) \leq \tilde{g}(x) \leq V(x)+bd, \quad \text{for } x \in E,$$

where $d = \frac{1}{\lambda}\sum_{n=0}^{\infty} na(n)$.

Remark: 1) If C be an atom,then $d = 1$;

2) If C is a ν_m-small set, then $d = \frac{m}{\nu_m(E)}$.

Comparison with Meyn & Glynn (1996)

Under the same condition, Meyn & Glynn (1996) established the following upper bound

$$|\tilde{g}(x)| \le c[V(x) + 1]$$

for some implicit constant c. Our bound presents an explicit representation of the constant c

$$|\tilde{g}(x)| \le \max\left\{\frac{b}{\inf_{x \in E} f(x)}, 1\right\}[V(x) + bd].$$

Key ideas in proof

- ▷ Construct a split chain and obtain the solution of Poisson equation

$$\tilde{g}(x) = \mathbb{E}_x\left[\sum_{j=0}^{\tau-1} \bar{g}(X_j)\right].$$

- ▷ Apply the comparison theorem and properties of the split chain to bound $\tilde{g}(x)$.
- ▷ A detailed analysis of the bounds on $\mathbb{E}_x\left[\sum_{j=0}^{\tau-1} I_C(X_j)\right]$.

Application: truncation approximation

Let V be a coercive function such that a sublevel set $A_n := \{x : V(x) \leq n\}$ is either empty or compact. Let $N(C) = \min\{n : C \subseteq A_n\}$ for a set C.

Corollary 1 Suppose that there exists a petite set C such that $\{_{(n)}P,\ n \geq N(C)\}$ uniformly satisfies D(V,f,b,C). Then we have

$$\left\| \pi - {}_{(n)}\pi \right\|_f \leq b[n + d(b+1)]\pi(A_n^c) + \int_{y \in A_n^c} \pi(dy)V(y).$$

Moreover, if $\pi(V) < \infty$, then $\left\| \pi - {}_{(n)}\pi \right\|_f \to 0$ as $n \to \infty$.

Application: asymptotic variance

Corollary 2 Suppose that D(V,f,b,C) holds for a petite set C and that $\pi(V^2) < \infty$. Then for any measurable function g with $0 \leq g \leq f$, we have

$$\begin{aligned} \gamma_g^2 &:= \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_\pi[(S_n(\bar{g}))^2] \\ &= 2\mathbb{E}_\pi[\tilde{g}(\Phi_0)\overline{g}(\Phi_0)] - \mathbb{E}_\pi[\overline{g}^2(\Phi_0)] \\ &\leq 2Cov_\pi(V(\Phi_0), g(\Phi_0)) - \mathbb{E}_\pi[\overline{g}^2(\Phi_0)]. \end{aligned}$$

If $\gamma_g^2 > 0$, then a CLT holds.

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Stochastically ordered Markov Chains

Let Φ_n be a DTMC on the state space $E = \mathbb{R}^+$ or $E = \mathbb{Z}^+$. Let Φ_n^1 and Φ_n^2 be two copies of Φ_n with different initial states and the same transition kernel.

The chain $\{\Phi_k, k \geq 0\}$ is called stochastically ordered if Φ_n^1 is stochastically larger than Φ_n^2 ($P(\Phi_n^1 \leq z) \leq P(\Phi_n^2 \leq z)$, $z \in E$) for any $n \geq 1$, whenever Φ_0^1 is stochastically larger than Φ_0^2.

Monotonicity for a solution

Suppose that the state 0 is accessible, then

$$\tilde{g}_0(x) = \mathbb{E}_x\left[\sum_{k=0}^{\tau_0-1} \overline{g}(\Phi_k)\right], \ x \in E,$$

is a solution of Poisson's equation.

Theorem 2 (Liu & L. 2023)

Let Φ_n be a Harris positive recurrent and stochastically ordered Markov chain with invariant distribution π. If the state 0 is accessible from any other state, $\boldsymbol{g}(x)$ is non-decreasing in x and $\pi(|\boldsymbol{g}|) < \infty$, then $\tilde{g}_0(x)$ is also non-decreasing.

Key ideas in proof

- Stochastic ordering is distributionally equivalent to pathwise ordering in the sense that

$$\Phi_n^1(\omega^1) \geq \Phi_n^2(\omega^2) \text{ if } \Phi_0^1(\omega^1) \geq \Phi_0^2(\omega^2).$$

- Note $\overline{g}(x) = \boldsymbol{g}(x) - \pi(\boldsymbol{g})$ is non-decreasing in x.
- For $x \geq y$

$$\tilde{g}_0(x) = \mathbb{E}_x\left[\sum_{k=0}^{\tau_0^1-1} \overline{g}(\Phi_k^1)\right] \geq \mathbb{E}_y\left[\sum_{k=0}^{\tau_0^2-1} \overline{g}(\Phi_k^2)\right] = \tilde{g}_0(y).$$

Comparison

Comparison with Glynn & Infanger (2022):

(1) For $E = \mathbb{Z}^+$, both results are the same. For $E = \mathbb{R}^+$, the results differ in the assumptions.

(2) Different arguments.

Block-Structured Markov Chains

Let $\{(X_k, Y_k), k \geq 0\}$ be a block-structured two-dimensional Markov chain with a countable state space $E = \bigcup_{i=0}^{\infty} \ell(i)$, where

$$\ell(i) := \{(i,j), i \geq 0, 1 \leq j \leq d\}$$

denotes the level set. These chains are useful for modelling the phase-type queues. (Neuts 1988, Latouche and Ramaswami 1998.)

Let $\boldsymbol{P} = (P(k,i;l,j))_{(k,i),(l,j)\in E}$ be the transition probability matrix of the chain $\{(X_k, Y_k), k \geq 0\}$.

Block-Monotone Markov chains (BMMC)

A stochastic matrix $\boldsymbol{P}$ is called stochastically block-monotone with block size $d \geq 1$ if for all $k, \ell \geq 0$,

$$\sum_{m=\ell}^{\infty} P(k,i;m,j) \leq \sum_{m=\ell}^{\infty} P(k+1,i;m,j), \quad 1 \leq i,j \leq d.$$

A function $\boldsymbol{g} = g(i,j)_{(i,j)\in E}$ is called block non-decreasing with block size d if for all $i \geq 1$, $1 \leq j \leq d$, we have $g(i,j) \leq g(i+1,j)$.

For a fixed state $(\alpha', \alpha'') \in E$,

$$\tilde{g}_{(\alpha',\alpha'')}(k,i) = \mathbb{E}_{(k,i)}\left[\sum_{k=0}^{\tau_{(\alpha',\alpha'')}-1} \overline{g}(X_k, Y_k)\right], \quad (k,i) \in E,$$

is one solution of Poisson's equation.

Theorem 3 (Liu & L. 2023)

Suppose that $\{(X_k, Y_k), k \geq 0\}$ is an irreducible and positive recurrent BMMC. If the forcing function g is block non-decreasing with size d and satisfies $\pi(|g|) < \infty$, then for any state $(\alpha', \alpha'') \in E$, the solution $\tilde{\boldsymbol{g}}_{(\alpha',\alpha'')}$ is also block non-decreasing with size d.

Key ideas of proof

▷ Our arguments are based on the sample path analysis. We first consider the case of $(\alpha', \alpha'') = (0, 1)$.

▷ Masuyama (2015) shows that a block-monotone Markov Chain, is pathwise ordered in the first variable X_k given the same second phase variable.

▷ For the general case of (α', α''), we know that

$$\tilde{\boldsymbol{g}}_{(\boldsymbol{\alpha}',\boldsymbol{\alpha}'')} - \tilde{\boldsymbol{g}}_{(\mathbf{0},\mathbf{0})} = \tilde{g}_{(\alpha',\alpha'')}(0, 1)\boldsymbol{e}.$$

An illustrative example

We consider a quasi-birth-and-death (QBD) process with transition matrix $\boldsymbol{P}$ given by

$$\boldsymbol{P} = \begin{pmatrix} A_{-1}+A_0 & A_1 & 0 & 0 & \cdots \\ A_{-1} & A_0 & A_1 & 0 & \cdots \\ 0 & A_{-1} & A_0 & A_1 & \cdots \\ 0 & 0 & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Obviously P is block-monotone.

Here, we let

$$A_{-1} = \begin{pmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0 & 0.3 \\ 0.3 & 0 & 0.2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0 \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.1 & 0 \\ 0 & 0.1 & 0.1 \end{pmatrix}.$$

Now, take $g(i,j) = i + \frac{1}{j+1}, i \geq 0, j \in \{1,2,3\}$. Based on Liu, L. & Zhao (2023), we obtain $\pi(|g|) \approx 1.2658 < \infty$ and the solution $\tilde{\boldsymbol{g}}_{(\mathbf{0},\mathbf{1})}$.

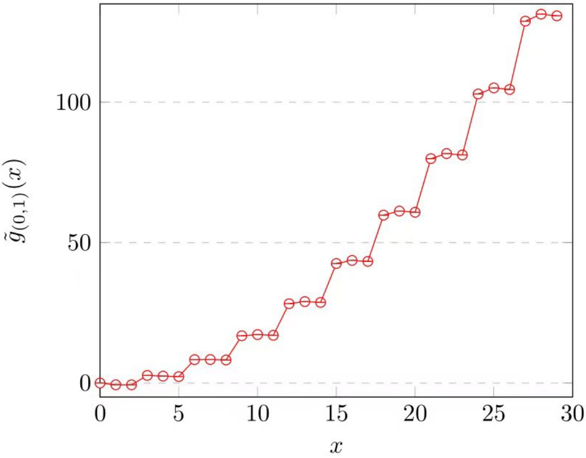

Figure: The values of the solution $\tilde{\boldsymbol{g}}_{(\mathbf{0},\mathbf{1})}$

Non-recurrent BMMCs

For a non-recurrent BMMC and a finite non-negative function $\boldsymbol{g}$, Poisson's equation is written as

$$(\boldsymbol{I} - \boldsymbol{P})\tilde{\boldsymbol{g}} = \boldsymbol{g}.$$

The function $\tilde{g}_\infty$, defined by

$$\tilde{g}_\infty(k, i) = \mathbb{E}_{(k,i)}\left[\sum_{k=0}^{\infty} g(X_k, Y_k)\right], \quad (k, i) \in E,$$

is the minimal non-negative solution of Poisson's equation.

Block monotonicity

Theorem 4 (Liu & L. 2023)

Let $\{(X_k, Y_k), k \geq 0\}$ be an irreducible and non-recurrent BMMC. If the forcing function $\boldsymbol{g}$ is block non-increasing with size d, then the solution $\tilde{\boldsymbol{g}}_\infty$ is also block non-increasing with size d.

Remark 2. In fact, if the forcing function $\boldsymbol{g}$ is block non-decreasing, the solution $\tilde{\boldsymbol{g}}_\infty$ is also block non-decreasing. However, in this case, the solution function $\tilde{\boldsymbol{g}}_\infty = \infty$.

Concluding remarks

(1) The results for bound and monotonicity of the Poisson equation can be expected to hold for continuous-time Markov processes.

(2) It is interesting to extend the concept of block-monotonicity to Markov switching fluid queue/ Brownian motion

Thank you for your attention!